# ALMOST EVERY INTERVAL TRANSLATION MAP OF THREE INTERVALS IS FINITE TYPE

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ABSTRACT. Interval translation maps (ITMs) are non-invertible generalizations of interval exchanges (IETs). The dynamics of finite type ITMs is similar to IETs, while infinite type ITMs are known to exhibit new interesting effects. In this paper, we prove the finiteness conjecture for the ITMs of three intervals. Namely, the subset of ITMs of finite type is open, dense, and has full Lebesgue measure. For this, we show that any ITM of three intervals can be reduced either to a rotation or to a double rotation.

### 1. Introduction and main result

1.1. Interval translation maps. Let  $\Omega \subset \mathbb{R}$  be a semi-interval split into d disjoint semi-intervals,  $\Omega = \sqcup_{j=1}^d \Delta_j$ ,  $\Delta_j = [\beta_{j-1}, \beta_j)$ . An interval translation  $T \colon \Omega \to \Omega$  is a map given by a translation on each of  $\Delta_j$ ,  $T|_{\Delta_j} \colon x \mapsto x + \gamma_j$ , the vector  $(\gamma_1, \ldots, \gamma_d)$  is fixed. After normalization  $\Omega := [0,1)$ , the space ITM(d) of d intervals' translations is a convex polytope in  $\mathbb{R}^{2d-1}$ . We endow it with the Euclidean metric and the Lebesgue measure. An example of an ITM is shown on Figure 1.1. We draw  $\Omega$  as a horizontal line, the splitting of  $\Omega$  into  $\Delta_j$  as arcs of different styles atop of the line, and the images of  $\Delta_j$  as the mirrored arcs below the line.



FIGURE 1.1. An interval translation map of 3 intervals.

Interval translation maps (ITMs) were first introduced in 1995 by Boshernitzan, Kornfeld in [1]. They are a generalization of interval exchange transformations. Unlike IETs the ITMs are generally not invertible.

Define  $\Omega_0 = \Omega$ ,  $\Omega_n = T\Omega_{n-1}$  for  $n \geq 1$ , and let X be the closure of  $\bigcap_{n=1}^{\infty} \Omega_n$ . An interval translation map is called of *finite type* if  $\Omega_{n+1} = \Omega_n$  for some n, otherwise it is called of *infinite type*. Denote the set of infinite type ITMs by S.

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Schmeling and Troubetzkoy [10] proved that for any d, an ITM is of finite type iff X consists of a finite union of intervals. In this case, some power of the restriction  $T|_X$  is an interval exchange transformation. If T is of infinite type and  $T|_X$  is transitive, then X is a Cantor set. Additionally, Bruin and Troubetzkoy [4] showed that if X is transitive, then it is minimal i.e. every orbit is dense.

In their pioneering paper [1], Boshernitzan and Kornfeld gave the first example of an ITM of infinite type. They also asked, "To what extent the ITMs of finite type are typical? In particular, is it true that 'almost all' ITMs are of finite type?" Then they remarked that the answer is affirmative for d=3 but gave no proof of that.

Until now, the question was answered (positively) only for some specific families of ITMs: by Bruin, Troubetzkoy [4] for a 2-parameter family in ITM(3), by Bruin [2] for a d-parameter family in ITM(d+1), and by Suzuki, Ito, Aihara [12] and Bruin, Clack [3] for so-called double rotations, see Subsection 1.3. Schmeling, Troubetzkoy [10] established a number of related topological results, in particular, that the interval translation mappings of infinite type form a  $G_{\delta}$  subset of the set of all interval translation mappings, whereas the interval translation mappings of finite type contain an open subset.

But the original question stood open. In this paper, we finally justify the affirmative answer for general ITMs of three intervals:

**Theorem 1.1.** In the space ITM(3), the set  $S \cap ITM(3)$  has zero Lebesgue measure.

Remark 1.2. The numerical estimate by [3] infers that its Hausdorff dimension H satisfies  $4 \le H \le 4.88$ .

The same question about d > 3 intervals remains open.

The entropy and word growth properties of piecewise translations, including their higher dimension generalizations, was studied in [7], [8], [5], [9]. The question of their (unique) ergodicity was considered by [4], [6], [3].

We always assume  $\gamma_i \neq 0$  and  $\beta_i - \beta_{i-1} \neq 0$  for all  $1 \leq i \leq d$ .

**1.2.** Tight ITMs. Let T be any interval translation map. We say that an interval  $\Delta \subset \Omega$  is T-regular (or simply regular) if there exists  $N \in \mathbb{N}$  such that for any  $x \in [0,1)$  there exists  $1 \leq n < N$  such that  $T^n x \in \Delta$ . In particular, every point of  $\Delta$  returns to  $\Delta$  after a uniformly bounded number of iterates of T, so the induced (i.e. first-return) map of T is well defined on  $\Delta$ . We denote the induced map by  $T_{\Delta}$ . We say that an interval  $\Delta \subset \Omega$  is a trap if it is regular and  $T\Delta \subset \Delta$ . In this case, we have  $T_{\Delta} = T|_{\Delta}$ .

The following Lemma shows that the properties to be finite or infinite type are preserved by inductions.

**Lemma 1.3.** Assume  $T|_X$  is transitive. Then,

- i) if there exists a regular  $\Delta$  such that  $T_{\Delta}$  has finite type, then T has finite type;
- ii) if T is finite type, then for any regular  $\Delta$  the map  $T_{\Delta}$  has finite type.

*Proof.* First we prove i. Assume T has infinite type. Then X is a T-invariant Cantor set with minimal dynamics (see [10], [4]). Because  $\Delta$  is regular,  $Y = X \cap \Delta$  is a nonempty Cantor set. It is also  $T_{\Delta}$ -invariant, and thus  $Y \subset X(T_{\Delta})$ . Because  $T_{\Delta}$  has finite type,  $X(T_{\Delta})$  is a finite union of intervals with an IET on them. This implies  $Y \neq X(T_{\Delta})$ . Because X is transitive, it is minimal for T, and thus  $X(T_{\Delta})$  is minimal for  $T_{\Delta}$ . But we have just shown that it contains another invariant closed set. This contradiction proves the lemma.

Recall that X is a finite union of intervals with an IET on them. So, because X is transitive, it is also minimal.

Now we prove ii. Assume  $T_{\Delta}$  has infinite type. Then  $X(T_{\Delta})$  is a  $T_{\Delta}$ -invariant transitive set. Denote by Y the union of its N iterates by T:

$$Y = \bigcup_{n=1}^{N} T^n X(T_{\Delta}).$$

Now Y is a T-invariant closed set and  $Y \subseteq X$ . This contradicts the minimality of X and thus proves the lemma.

For any  $M \subset [0,1]$ , we denote  $[M) := [\inf M, \sup M)$ . We say that an interval translation map  $T: \Omega \to \Omega$  is tight if  $[T\Omega) = [\Omega]$ . We denote the space of tight interval translation maps of d intervals by TITM(d). Recall that ITM(d) is a filled convex polytope in  $\mathbb{R}^{2d-1}$ , and observe that TITM(d) is a connected finite union of filled convex polytopes in  $\mathbb{R}^{2d-3}$ .

It is easy to see that for any  $T \in ITM(d)$ , there exists a trap  $\Delta$  such that the map  $T_{\Delta}$ is a tight interval translation map of r intervals,  $r \leq d$ . Namely, take  $X_0' := \Omega$ , and let  $X_k' := [TX_{k-1}'), \ \Delta := \bigcap_{k=0}^{\infty} X_k'$ . Then  $\Delta \supset X$ ,  $\Delta$  is a trap, and  $T_{\Delta}$  is tight.

However, to prove Theorem 1.1, we will need more explicit way to produce such a  $\Delta$ .

For d=2, we have  $\Delta=[\beta_1+\gamma_2,\beta_1+\gamma_1)$ , see Figure 1.2 and keep in mind that  $\gamma_1>0$ ,  $\gamma_2 < 0$ . Indeed, for any  $x < \beta_1$  we have  $Tx = x + \gamma_1$ . Thus there exists  $1 \le n_1 \le \left| \frac{\beta_1}{\gamma_1} \right| + 1$ 

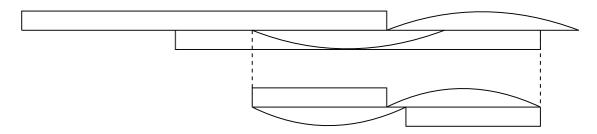


FIGURE 1.2. ITM of two intervals: rotation.

such that  $T^{n_1}x \in [\beta_1, \beta_1 + \gamma_1)$ . Similarly, for any  $x \geq \beta_1$  there exists  $1 \leq n_2 \leq \left[\frac{1-\beta_1}{|\gamma_2|}\right] + 1$  such that  $T^{n_2}x \in [\beta_1 + \gamma_2, \beta_1)$ . Thus  $\Delta$  is regular. Now note that  $T|_{\Delta}$  is just a rotation  $x \mapsto x + \gamma_1 \mod \Delta$ , so  $T\Delta = \Delta$ . Thus  $\Delta$  is a trap and  $T_{\Delta}$  is tight.

To obtain a similar formula for  $d \geq 3$ , consider two cases:

(1.1) 
$$\Delta_1 \cap T\Omega = \emptyset \text{ or } \Delta_d \cap T\Omega = \emptyset;$$

(1.2) 
$$\Delta_1 \cap T\Omega \neq \emptyset \text{ and } \Delta_d \cap T\Omega \neq \emptyset.$$

In the first case, we can completely remove  $\Delta_1$  or  $\Delta_d$  and thus reduce the problem to the study of ITM(d-1). For ITM(3), this case can be ignored because then the map is necessary a rotation, and thus finite type.

In the second case, let  $I_{-}$  be the set  $\{i \mid \gamma_{i} < 0\}$  and  $I_{+}$  be the set  $\{i \mid \gamma_{i} > 0\}$ . Because  $\gamma_{1} > 0$  and  $\gamma_{d} < 0$ , the both sets are nonempty. Take the interval

$$\Delta = [\delta_0, \delta_1) = [\min_{i \in I_-} (\beta_{i-1} + \gamma_i), \max_{i \in I_+} (\beta_i + \gamma_i)),$$

see Figure 1.3. In the same way as for d=2 one can show that for any  $x<\delta_0$  or  $x>\delta_1$ 

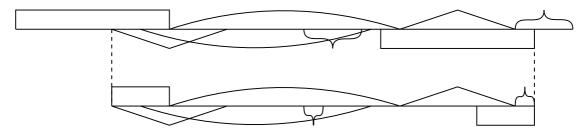


FIGURE 1.3. Fitting operator.

there exists a bounded n such that  $T^n x \in \Delta$ . Now, for any  $x \in \Delta$ , x either moves left or right. Assume it moves left. Then  $\min_{i \in I_-} (\beta_{i-1} + \gamma_i) \leq Tx < x$  which implies  $Tx \in \Delta$ . Similarly, for x moving right we have  $x < Tx \leq \max_{i \in I_+} (\beta_i + \gamma_i)$ . Thus  $Tx \in \Delta$ , and  $\Delta$  is a trap.

Because we are in the second case, for every  $2 \le i \le d$ , the left end  $\beta_{i-1}$  of  $\Delta_i$  belongs to  $\Delta$ . In particular, this is true for the  $i = \operatorname{argmin}_{i \in I_-}(\beta_{i-1} + \gamma_i)$ . So the left end of  $\Delta_i$  maps to  $\delta_0$ . Thus  $T_{\Delta}$  is tight from the left. Similarly, we show  $T_{\Delta}$  is tight from the right and thus tight in general.

Rescale the map  $T_{\Delta}$  so that  $\Delta$  becomes [0,1). We say that the result,  $\tilde{T}_{\Delta}$ , is the *fitting* of T and denote the *fitting operator*  $T \mapsto \tilde{T}_{\Delta}$  by  $\mathcal{F}_d$ :  $\mathrm{ITM}(d) \to \mathrm{TITM}(d)$ . When the value of d is clear, we will write  $\mathcal{F}$  instead of  $\mathcal{F}_d$  for brevity.

**Proposition 1.4.** In the case (1.2), the fitting operator is a piecewise rational map of the maximal rank 2d - 3.

*Proof.* We partition the space ITM(d) into the union of the cells  $C_{jk}$ ,  $j \neq k$ :

$$C_{jk} = \{ T \in \mathrm{ITM}(d) \mid j = \mathrm{argmin}_{i \in I_{-}}(\beta_{i-1} + \gamma_i), k = \mathrm{argmax}_{i \in I_{+}}(\beta_i + \gamma_i) \}.$$

The fitting operator is the composition of truncation and rescaling:

$$\mathcal{F} = \mathcal{R} \circ \mathcal{T}$$
.

The rescaling part is a rational map  $\mathcal{R}: \mathbb{R}^{2d-1} \to \mathbb{R}^{2d-3}$  of rank 2d-3. To understand the truncation part  $\mathcal{T}$ , we introduce new coordinates  $B_{i-1} = \beta_{i-1} + \gamma_i$ ,  $i = 1, \ldots, d$ , which are the images of the left ends of  $\Delta_i$ . In particular,  $B_0 = \gamma_0$ . At every cell  $C_{jk}$ , the truncation is a linear map  $\mathcal{T} \colon \mathbb{R}^{2d-1} \to \mathbb{R}^{2d-1}$ . Let us show it is invertible. In the coordinates  $(\beta_i, B_i)$ , the truncation has the form

(1.3) 
$$\begin{bmatrix} 0, \beta_{1}, \beta_{2}, \dots, \beta_{d-2}, \beta_{d-1}, 1 \\ B_{0}, B_{1}, \dots, B_{j-1}, \dots, B_{k}, \dots, B_{d-2}, B_{d-1} \end{bmatrix} \\ \mathcal{T} \\ \begin{bmatrix} B_{j-1}, \beta_{1}, \beta_{2}, \dots, \beta_{d-2}, \beta_{d-1}, B_{k} \\ B_{j-1} + B_{0}, B_{1}, \dots, B_{j-1}, \dots, B_{k}, \dots, B_{d-2}, B_{k} + B_{d-1} - \beta_{d-1}. \end{bmatrix}$$

Note that most of the coordinates are mapped identically. The non-identical part of (1.3)

(1.4) 
$$\begin{bmatrix} 0, \beta_{d-1}, 1 \\ B_0, B_{j-1}, B_k, B_{d-1} \end{bmatrix}$$

$$\downarrow \mathcal{T}$$

$$\begin{bmatrix} B_{j-1}, \beta_{d-1}, B_k \\ B_{j-1} + B_0, B_{j-1}, B_k, B_k + B_{d-1} - \beta_{d-1} \end{bmatrix}$$

which is clearly invertible. So, at every cell  $C_{ik}$  the fitting  $\mathcal{F}$  is a rational map of rank 2d - 3.

All the previous considerations are valid for any number of intervals. In what follows, we have the proofs only for the case of d=3.

**Double rotations.** Following Suzuki, Ito, Aihara [12], we introduce the family of double rotations. A double rotation is a map  $f_{(a,b,c)}:[0,1)\to[0,1)$  defined by

$$f_{(a,b,c)}(x) = \begin{cases} \{x+a\}, & \text{if } x \in [0,c), \\ \{x+b\}, & \text{if } x \in [c,1). \end{cases}$$

In the circle representation  $[0,1) \equiv S^1$ , a double rotation is a map  $S^1 \to S^1$  defined by independent rotations of two complementary arcs of  $S^1$ . Clearly, any double rotation is an ITM of two to four intervals. Denote the space of parameters (a, b, c) of double rotations by  $Rot(2) = [0,1) \times [0,1) \times [0,1)$ . In the same way we define the family Rot(n)of n-rotations.

For the double rotations, the finiteness problem is solved by Bruin and Clack in [3] using the renormalization operator of Suzuki, Ito, Aihara [12]. Namely, they proved the following

**Theorem 1.5** (Bruin, Clack, 2012). In the space Rot(2), the set  $\mathcal{R} = \mathcal{S} \cap Rot(2)$  of ITMs of infinite type has zero Lebesgue measure.

In the present paper we show that the finiteness problem for ITM(3) reduces to the one for the double rotations.

**Theorem 1.6.** The space TITM(3) splits into countably many open sets  $A, A', B, B_i, C_i$ ,  $i \in \mathbb{N}$ , such that their union U is dense in TITM(3), and the complement  $K = \text{TITM}(3) \setminus U$  which is a union of countably many hyperplanes. Moreover,

- any  $T \in A \cup A'$  is a double rotation,
- any  $T \in B$  is reduced to a double rotation via a single Type 1 induction,
- for any  $i \in \mathbb{N}$ , any  $T \in B_i$  is reduced to a double rotation via a single Type 2 induction.
- for any  $i \in \mathbb{N}$ , any  $T \in C_i$  is reduced to a single rotation via a single Type 2 induction.

On every piece  $A, A', B, B_i, C_i$ , these inductions are invertible rational maps.

Conjecture 1.7. The space TITM(d) splits into countably many open sets  $A_i$  such that their union U is dense in TITM(d), and the complement  $K_d = \text{TITM}(d) \setminus U$  which is a union of countably many hyperplanes. Any  $T \in U$  is reduced to an (d-1)-rotation via some induction.

Because the inductions are local diffeomorphisms, the preimages of zero measure sets are zero measure sets. The Hausdorff dimension is also preserved. This allows to transfer the results of [12] and [3] to TITM(3).

#### 2. Proof of Theorem 1.6

Let  $T \in \text{TITM}(3)$  with  $\Omega = [0, 1)$ . Without loss of generality, we can assume  $\gamma_1 > 0$  and  $\gamma_3 < 0$ . Because T is tight, some interval (not  $\Delta_1$ ) must go to the leftmost position, and some interval (not  $\Delta_3$ ) must go to the rightmost position. Obviously, there are 3 cases:

$$\begin{array}{cccc} & A & A' & B\&C \\ Leftmost & \Delta_2 & \Delta_3 & \Delta_3 \\ Rightmost & \Delta_1 & \Delta_2 & \Delta_1 \end{array}$$

The cases A and A' are mirror images of each other, so we consider only case A of these two.

- **2.1.** Case A. Double rotation in disguise. In this case,  $\Delta_2$  goes to the leftmost position and  $\Delta_1$  goes to the rightmost position, see Figure 2.1. Then T is a double rotation with  $c = \beta_2$  (i.e. the first arc is  $\Delta_1 \cup \Delta_2$  and the second one is  $\Delta_3$ ) and  $a = -|\Delta_1|$ ,  $b = \gamma_3$ .
- **2.2.** Cases B and C. Induction. In this case,  $\Delta_1$  goes to the rightmost position and  $\Delta_3$  goes to the leftmost position. Because of the symmetry, we can assume without loss of generality that  $|\Delta_1| \geq |\Delta_3|$ . Consider the two sub-cases:  $\gamma_2 < 0$  and  $\gamma_2 > 0$ .

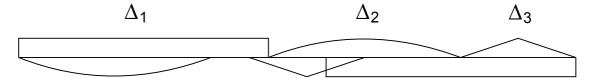


FIGURE 2.1. Double rotation in disguise.

Sub-Case  $\gamma_2 < 0$  [piece B].

**Proposition 2.1.** In this case,  $\Delta = \Delta_1 \cup \Delta_2$  is regular with the return time  $\leq 2$ .  $T_{\Delta}$  is a tight ITM of three intervals which is a double rotation.

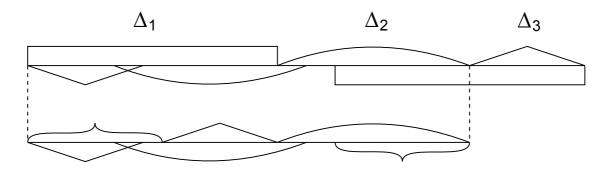


FIGURE 2.2. Induction to  $\Delta_1 \cup \Delta_2$ .

*Proof.*  $\gamma_2 < 0$  implies  $T\Delta_2 \subset \Delta$ , so  $\Delta_2$  returns to  $\Delta$  in one piece after a single iteration of T. On the other hand,  $\Delta_1$  is split by the first return map into two pieces,  $\Delta'_1$  and  $\Delta''_1$ , separated by the point  $\beta_2 - \gamma_1$ , see Figure 2.2. For the first piece,

$$T\Delta_1' = T\Delta_1 \cap \Delta \subset \Delta.$$

For the second piece, we have

$$T\Delta_1'' = T\Delta_1 \setminus \Delta = \Delta_3$$
, and  $T^2\Delta_1'' = T\Delta_3 \subset \Delta_1 \subset \Delta$ .

Here we used  $|\Delta_1| \geq |\Delta_3|$ . We have just shown that  $\Delta$  is regular.  $T_{\Delta}$  is a tight ITM of three intervals,  $\Delta'_1, \Delta''_1, \Delta_2$ . Note that  $T_{\Delta}$  sends  $\Delta'_1$  to the rightmost position and  $\Delta''_1$  to the leftmost one. Thus we have Case A for  $T_{\Delta}$ , and  $T_{\Delta}$  is a double rotation.

Note that the induction operator  $T \mapsto T_{\Delta}$  is an invertible rational map on B. 

Remark 2.2. The sub-case  $\gamma_2 > 0$  is a generalized Bruin-Troubetzkoy family [4] with one extra degree of freedom,  $\gamma_2$ .

Sub-Case  $\gamma_2 > 0$  [pieces  $B_i$ ,  $C_i$ ].

**Proposition 2.3.** In this case,  $\Delta = \Delta_2 \cup \Delta_3$  is regular, and  $T_{\Delta}$  is a tight ITM of three intervals which is a double rotation.

*Proof.* Similarly,  $\gamma_2 > 0$  implies  $T\Delta_2 \subset \Delta$ , so  $\Delta_2$  returns to  $\Delta$  in one piece after a single iteration of T. Let us now observe the return of  $\Delta_3$ .

We know  $T\Delta_3$  is at the leftmost position, which implies  $T\Delta_3 \subset \Delta_1$ . Thus  $T^2\Delta_3 =$  $T\Delta_3 + \gamma_1$ . Moreover, for any  $n \in \mathbb{N}$  such that  $T^n\Delta_3 \subset \Delta_1$ , we have  $T^n\Delta_3 = T\Delta_3 +$  $(n-1)\gamma_1$ . Let n be the maximal n such that  $T^2\Delta_3,\ldots,T^n\Delta_3\subset\Delta_1$ . There are two possibilities:

- i)  $T^{n+1}\Delta_3 \subset \Delta$ , or ii)  $T^{n+1}\Delta_3 \cap \Delta \neq \emptyset$ ,  $T^{n+1}\Delta_3 \cap \Delta \neq \emptyset$ .

Possibility i corresponds to  $B_i$ , i = n. In this case,  $\Delta$  is obviously regular, and  $T_{\Delta} \in ITM(2)$ , see Figure 2.3. By [1],  $T_{\Delta}$  reduces to a (single) rotation.

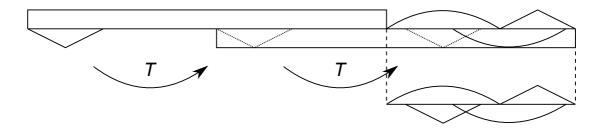


Figure 2.3. Induction to  $\Delta_2 \cup \Delta_3$ : possibility *i*.

Possibility ii corresponds to  $C_i$ , i = n. In this case,  $\Delta_3$  is split into two pieces  $\Delta_3'$  and  $\Delta_3''$ .  $\Delta_3''$  returns to  $\Delta$  after n+1 iterations of T and goes to the leftmost position, see Figure 2.4.  $\Delta_3'$  returns to  $\Delta$  after n+2 iterations and goes to the rightmost position. Thus we have Case A' for  $T_{\Delta}$ , and  $T_{\Delta}$  is a double rotation.

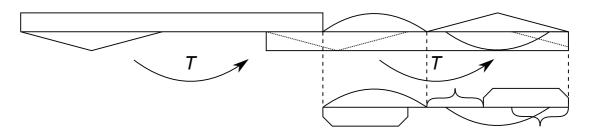


FIGURE 2.4. Induction to  $\Delta_2 \cup \Delta_3$ : possibility *ii*.

Note that the induction operator  $T \mapsto T_{\Delta}$  is an invertible rational map on each  $B_i$  or  $C_i$ .

# Proof of Theorem 1.1

Note that (apart from a few trivial exceptions) for any  $T \in ITM(3)$  the limit set X is either entirely periodic or transitive. All the infinite type ITMs belong to the latter case. Also for ITM(3) we can ignore the case (1.1), because in this case the ITM is just a rotation, and cannot be infinite type. Recall that by Lemma 1.3, the properties to be finite or infinite type are preserved by inductions.

Now we invoke Theorem 1.5 by Bruin and Clack [3]. Together with our Theorem 1.6 it implies that the set  $S \cap TITM(3)$  is a countable union of sets of zero Lebesgue measure. [3] also numerically estimated the Hausdorff dimension of the infinite type parameters to be between 2 and 2.88. Thus,

$$Leb(S \cap TITM(3)) = 0, \quad 2 \le dim(S \cap TITM(3)) \le 2.88.$$

By Proposition 1.4, the set  $S \cap ITM(3) = \mathcal{F}^{-1}(S \cap TITM(3))$  has zero Lebesgue measure and Hausdorff dimension between 4 and 4.88. The main theorem is proven.

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